69. Review of Multibody Charm Analyses

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69.1 Overview

The study of multibody charm decays is a vibrant field, with many new experimental results with implications reaching beyond charm, vast and fast-increasing datasets, and new developments in theory and experimental technique. This review is structured as follows. Sec. 69.2 summarises key elements of the theory of multibody charm decays. Sec. 69.3 describes model-independent approaches for precision measurements of mixing and *CP* violation and the critical role of charm threshold data in this context. Sec. 69.4 reviews applications of the techniques described, and their results. In Sec. 69.5, we conclude.

69.2 Kinematics & Models

The differential decay rate to a point $\mathbf{s} = (s_1, \ldots, s_n)$ in n dimensional phase space can be expressed as

$$d\Gamma = |\mathcal{M}(\mathbf{s})|^2 \left| \frac{\partial^n \phi}{\partial(s_1 \ldots s_n)} \right| d^n s \tag{69.1}$$

where $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ represents the density of states at $\mathbf{s}$, and $\mathcal{M}$ the matrix element for the decay at that point in phase space, which is $2, 5, 8, \ldots$ dimensional for D decays to $3, 4, 5, \ldots$ spinless particles. Additional parameters are required to fully describe decays involving particles with non-zero spin in the initial or final state.

For the important case of D decays to three pseudoscalars, the decay kinematics can be represented in a two-dimensional Dalitz plot [1]. This is usually parametrized in terms of $s_{12} \equiv (p_1 + p_2)^2$ and $s_{23} \equiv (p_2 + p_3)^2$, where p_1, p_2, and p_3 are the four-momenta of the final state particles. In terms of these variables, phase-space density is constant across the kinematically allowed region, so that any structure seen in the Dalitz plot is a direct consequence of the dynamics encoded in $|\mathcal{M}|^2$. Note that here, because the three-momenta of the decay products are confined to a plane, no parity violating kinematic observables can be constructed (unless they also violate rotational invariance). This is not the case for decays to four or more particles. These can therefore not be unambiguously described in terms of analogously-defined variables s_{ij}, s_{ijk}, which are parity-even. The use of parity-odd observables in four body decays is discussed below.

In the widely-used isobar approach, the matrix element $\mathcal{M}$ is modeled as a sum of interfering decay amplitudes, each proceeding through resonant two-body decays [2–6]. See Refs. [2, 7, 8] for a review of resonance phenomenology. In most analyses, each resonance is described by a relativistic Breit-Wigner [9] or Flatté [10] lineshape, and the model includes a non-resonant term with a constant phase and magnitude. This approach has well-known theoretical limitations, such as the violation of unitarity and analyticity, which can break the relationship between magnitude and phase across phase space. This motivates the use of more sophisticated descriptions, especially for broad, overlapping resonances (frequently found in S-wave components) where these limitations are particularly problematic. In charm analyses, these approaches have included the K-matrix approach [11] which respects two-body unitarity; the use of LASS scattering data [12]; dispersive methods [13–16]; multi-meson models using chiral Lagrangians [17–20] and methods based on chiral unitarity [21, 22]; QCD factorisation [23–26]; and quasi model-independent parametrizations which use generic lineshapes, with minimal theory input and many free parameters, for a subset of resonances [27–32]. An important example, with a rich resonance structure, is $D^0 \to K_S \pi^+ \pi^-$, which is a key channel in Charge-Parity (*CP*) violation and charm-mixing analyses. The first analysis by CLEO [33] described the Dalitz plot with approximately 5000 signal events with 10 resonant components.

This and analyses by Belle [34] and CDF [35] model the Dalitz plot as a sum of Breit Wigner and Flatté line shapes, and a non-resonant component. BaBar [36] and a more recent Belle analysis [37] on the other hand use a K-matrix description for the $\pi\pi$ S-wave based on [38] and input from LASS scattering data for the $K\pi$ S-wave, with no need to add a non-resonant component to describe the data. This approach is also followed in the latest analysis of this channel, published jointly by BaBar and Belle [39]. In total 18 resonant components, including four doubly Cabibbo suppressed ones, are required to describe the Dalitz plot with 1.1M $D^0 \to K_S\pi^+\pi^-$ events. Belle's and BaBar's data have been re-analyzed by [23] in a QCD factorization framework, using line-shape parametrizations for the S [40, 41] and P wave [14] contributions that preserve two-body unitarity and analyticity. The measurements give compatible results for the components they share.

The field of charm amplitude analyses remains very active. Publications since the last update of this review two years ago include Dalitz plot analyses of $D_s^+ \to K^+K^-\pi^+$, $K^+K_S\pi^0$, $K_S^0\pi^+\pi^0$, $\pi^+\pi^-\pi^+$, $D^0 \to K_S^0K^+K^-$ by BES III [31, 32, 42–44]; $D^0 \to K^-\pi^+\eta$ by Belle [45]; a re-analysis of BaBar's $D^0 \to K_SK^+K^-$ data in a factorisation approach [26] and of LHCb's $D^+ \to K^+K^-K^+$ data [46] in a chiral unitarity approach [22]; four-body amplitude analyses of $D_s^+ \to \pi^+\pi^+\pi^-\eta$, $K^-K^+\pi^+\pi^0$, $K_S^0K^-\pi^+\pi^+$ by BES III [31, 47–49] and of $D^0 \to K^+K^-\pi^+\pi^-$ by LHCb [50]; and the amplitude analysis of the charm baryon decay $\Xi_c^0 \to \Xi^0K^+K^-$ by Belle [51]. Noteworthy is the increasing sophistication of recent amplitude analyses, most of which go substantially beyond the isobar model with Breit Wigner and Flatté lineshapes. However, with the notable exceptions of [15, 16, 22, 46, 52], they remain within the isobar framework which describes the decay as a series of two-body processes, and ignores long-range hadronic effects such as re-scattering and does not respect three (or four)-body unitarity and analyticity.

Several groups work on improved models. Dispersive techniques, which respect three-body unitarity and analyticity, have been applied to regions of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K_S\pi^0\pi^+$ Dalitz plots below the $\eta'K$ threshold [15, 16], where they provide a good description of the data with fewer fit parameters than the isobar approach. Ref. [52] uses a unitary coupled channel approach to describe $D^+ \to K^-\pi^+\pi^+$, which has no restrictions on the kinematic range, but requires additional parameters to describe the Dalitz plot above the $\eta'K$ threshold. Using an effective chiral Lagrangian, the authors of Ref. [19] provide a description of the annihilation contribution to the decay amplitude which respects three-body unitarity. The approach provides a good description of LHCb $D^+ \to K^+K^-K^+$ data, with fewer parameters than an equivalent isobar model [46]. The same channel has more recently been re-analysed by the authors of [22], who argue that the internal emission diagram should dominate and use a chiral unitarity-based approach to achieve a reasonable description of the data with two free parameters.

Limitations in the theoretical description of interfering resonances are the leading source of systematic uncertainty in many analyses. This is set to become increasingly problematic given the statistical precision achievable with the vast, clean charm samples available at the B factories, LHCb, and their upgrades, and BES III. In some cases, the model uncertainty can be removed through model-independent methods, often relying on input from the charm threshold, as discussed below. The authors of [53] expand the scope and applicability of the quasi model-independent approach in amplitude fits. At the same time, increasingly sophisticated models are being developed, and applied to data. The authors of [20] and [54] provide valuable frameworks for sophisticated amplitude analyses.

69.3 Model Independent Methods and the Charm Threshold

The precision measurement of mixing or CP violation parameters such γ/ϕ_3 from multibody charm decays requires as input the phase-differences between the D^0 and $\overline{D}^0$ amplitudes across phase space, as well as their magnitudes, for each final state of interest. While the magnitudes are fairly easily measured, the phase information requires either amplitude models with reliable phase motion, or model-independent approaches.

Model-independent measurements of the relevant phase differences rely on interference effects in the decays of well-defined coherent superpositions of D^0 and $\overline{D}^0$. These are accessible at the charm threshold, where CLEO–c and BES III operate [55–63]. Charm mixing also results in a (time-dependent) $D^0 - \overline{D}^0$ superposition, that can be used to measure the relevant phase information as input to γ/ϕ_3 measurements. This method is particularly powerful in doubly Cabibbo-suppressed decays such as $D^0 \to K^+\pi^-\pi^+\pi^-$, and when used in combination with threshold data [64,65]. Under some circumstances, with large data sets, the relevant strong phases and γ/ϕ_3 can be extracted simultaneously without external input, for example in simultaneous analysis of the $B^0 \to DK^+\pi^-$ Dalitz plot and that of the subsequent $D \to K^0_S\pi^+\pi^-$ decay [66]. However, the global effort to achieve a measurement of γ/ϕ_3 to sub-degree precision will continue to rely critically on input from the charm threshold and BES III's increasing datasets [67]. A particularly noteworthy recent development is the substantial improvement not only in the precision on γ but also on charm mixing parameters achieved through a combined fit of charm mixing, charm threshold, $B \to DK$ and related data [68].

The model-independent phase information of the charm decay is provided either integrated over the entire phase space of the decay, or in sub-regions/bins. The results can be expressed in terms of one complex parameter $\mathcal{Z} = Re^{-i\delta} = c + is$ per pair of CP-conjugate bins, with magnitude $R \leq 1$. Larger R values lead to higher sensitivity to γ/ϕ_3. Amplitude models can be used to optimise the binning for sensitivity to γ/ϕ_3, without introducing a model-dependent bias in the result; novel unbinned approaches are expected to increase the precision on γ in future analyses [69].

CLEO–c data have been analyzed to provide binned $\mathcal{Z}$ for the self-conjugate decays $D^0 \to K_S\pi^+\pi^-$, $K_SK^+K^-$, $\pi^+\pi^-\pi^+\pi^-$, and $K_S\pi^-\pi^+\pi^0$ and phase space-integrated values for $D^0, \overline{D}^0 \to K_SK^+\pi^-$, $K^+\pi^-\pi^0$ and $K^+\pi^-\pi^+\pi^+$ [70–75]. BES III significantly improved the precision for the $K_S\pi^+\pi^-$, $K_SK^+K^-$, $K^+\pi^-\pi^0$ and $K^+\pi^-\pi^+\pi^-$ final states [76–79]. Critically, BES III was able to perform this measurement in four separate bins of $K^+\pi^-\pi^+\pi^-$ phase space. Studies based on recent $D^0 \to K^\pm\pi^\mp\pi^\pm\pi^\mp$ amplitude models [30] indicate that such a binned analysis could lead to the most precise individual γ measurement [80], especially when combined with LHCb charm mixing data [65, 75, 81]. For self-conjugate decays such as $D^0 \to \pi^+\pi^-\pi^0$, analysed with a single pair of bins, $\mathcal{Z}$ is real-valued, and usually expressed in terms of the CP-even fraction $F_+ \equiv \frac{1}{2}(\text{Re}(\mathcal{Z}) + 1)$, defined such that a CP-even eigenstate has $F_+ = 1$, while a CP-odd eigenstate has $F_+ = 0$ [61]. Recent analyses of CLEO–c data reveal that $D^0 \to \pi^+\pi^-\pi^0$ is compatible with being completely CP-even with $F_+ = 0.973\pm0.017$, while $D^0 \to K^+K^-\pi^0$ has $F_+ = 0.732\pm0.055$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$ has $F_+ = 0.769 \pm 0.023$ and $D^0 \to K_S\pi^+\pi^-\pi^0$ has 0.238 ± 0.020 [62, 72, 73].

It is interesting to compare these values with those obtained from amplitude models as a cross-check of the models' phase-motion. $F_+^{4\pi \text{ model}} = 0.729 \pm 0.020$ calculated from Ref. [29]'s $D^0 \to \pi^+\pi^-\pi^+\pi^-$ model compares well to the measured value given above, as does $\left|\mathcal{Z}^{K3\pi \text{ model}}\right| = 0.459 \pm 0.025$ [30] to $\mathcal{Z}^{K3\pi \text{ meas}} = 0.44^{+0.10}_{-0.09} \exp(-i(161^\circ{}^{+28^\circ}_{-18^\circ}))$ [75, 81]. Only for self-conjugate decays or multiple bins does the model provide relevant phase information. The latest bin wise comparisons between model and data for $D^0 \to K_S\pi^+\pi^-$, $K_SK^+K^-$, $\pi^+\pi^-\pi^+\pi^-$, and $K^\pm\pi^\mp\pi^\pm\pi^\mp$ can be found in [72, 76–79] and generally show good agreement, which is a welcome surprise given the preceding discussion on the theoretical shortcomings of such amplitude models.

69.4 Applications of multibody charm analyses

Amplitude analyses provide sensitivity to both relative magnitudes and phases of the interfering decay amplitudes. It is especially this sensitivity to phases that makes amplitude analyses such a uniquely powerful tool for studying a wide range of phenomena. Here we concentrate on their use for CP violation and mixing measurements in charm, and charm inputs to CP violation analyses in B meson decays (properties of light-meson resonances determined in D amplitude analyses are reported in the light-unflavored-meson section of this *Review*).

69.4.1 *Time-integrated searches for CP violation in charm*

Comparing the results of amplitude fits for CP-conjugate decay modes provides a measure of CP violation. Recent CP violation searches using this method include amplitude analyses of $D^0 \to K^0_S K^\pm \pi^\mp$ and $D^0 \to K^+K^-\pi^+\pi^-$ by LHCb [82, 83], and $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [29, 84] using CLEO data.

A widely-used amplitude model-independent technique to search for local CP violation is based on performing a χ^2 comparison of CP-conjugate phase-space distributions. This method was pioneered by BaBar [85] and developed further in [86–88], with results reported by BaBar [85, 89] and LHCb in $D^\pm \to K^+K^-\pi^\pm$ [90, 91], CDF in $D^0 \to K_S\pi^+\pi^-$ [35], and LHCb in $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $\Xi_c^+ \to pK^-\pi^+$ [88, 92, 93]. Unbinned methods can increase sensitivity [94] and have been applied by LHCb to $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to \pi^+\pi^-\pi^0$, $D^0 \to \pi^+\pi^-\pi^+\pi$, and $\Xi_c^+ \to pK^-\pi^+$ [92, 93, 95, 96].

An alternative model-independent approach is based on observables in four body decays that are odd under motion reversal ("naïve T") [97–105], which is equivalent to P for scalar particles [105]. One such observable is $C_T = \vec{p}_2 \cdot (\vec{p}_3 \times \vec{p}_4) = (1/m_D)\epsilon_{\alpha\beta\gamma\delta}\, p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, where $\vec{p}_i$ are the decay products' three momenta in the decay's rest frame, and p_i are their four-momenta. Identical particles (as in $D^0 \to K^+\underline{\pi}^-\pi^+\underline{\pi}^-$) are ordered by momentum magnitude. Comparing the P violating asymmetry $A_T \equiv \frac{\Gamma(C_T>0)-\Gamma(C_T<0)}{\Gamma(C_T>0)+\Gamma(C_T<0)}$ with its C-conjugate provides sensitivity to CP violation. Searches for CP violation in this manner have been carried out by BaBar, Belle, FOCUS and LHCb in $D^0 \to K^+K^-\pi^+\pi^-$ [106–109], $D^+ \to K^+K_S\pi^+\pi^-$, $D_s^+ \to K^+K_S\pi^+\pi^-$ [110], $D^0 \to K_S\pi^+\pi^-\pi^0$ [111], and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [96]. The authors of [109] use additional P-odd variables and [108, 111]/[96] increase sensitivity by resolving four-body phase space in a binned/unbinned approach.

The results of all measurements described in this section are compatible with CP conservation in charm. Given the discovery of CP violation in $D^0 \to K^+K^-$, $D^0 \to \pi^+\pi^-$ decays [112], and the vast data samples about to be collected, one might expect this to change in the foreseeable future.

69.4.2 *CP violation in decays of Beauty to Charm*

Neutral D mesons originating from $B^- \to DK^-$ (here denoted as D_{B^-}) are a superposition of D^0 and $\overline{D}^0$ with a relative phase that depends on the CKM unitarity triangle parameter γ/ϕ_3,

$$D_{B^-} \propto D^0 + r_B e^{i(\delta_B - \gamma)}\overline{D}^0,$$

where δ_B is a CP conserving strong phase, and $r_B \sim 0.1$. In the corresponding CP-conjugate expression, γ/ϕ_3 changes sign. An amplitude analysis of the subsequent decay of the $D_{B^\pm}$ to a state accessible to both D^0 and $\overline{D}^0$ allows the measurement of γ/ϕ_3 [56, 113–116]. The method generalizes to similar B hadron decays, such as $B^0 \to DK^{*0}$. Measurements based on this technique have been reported by BaBar [117, 118], Belle [34, 119] and LHCb [120–129]. The most precise individual result with an uncertainty of $\sim 5^\circ$ comes from an amplitude model-independent study of $D_{B^-} \to K_S\pi^+\pi^-$ and $D_{B^-} \to K_SK^+K^-$ [130], using input from the charm threshold [70, 77] as described above. Combining measurements in multiple decay modes from LHCb, CLEO–c and BES III in an amplitude-model independent approach leads to a current uncertainty on γ/ϕ_3 of 4° [68, 131].

The interference between mixing and decay in $B^0 \to D^0h^0$ with $h^0 = \pi^0, \eta, \omega$ provides sensitivity to β, which can be extracted from the Dalitz plot of the subsequent $D^0 \to K_S\pi^+\pi^-$ decay [39, 132–135]. The combined BaBar/Belle analysis based on this technique resolved the ambiguity in β present in other measurements, such as $B^0 \to J/\psi K_S$, in favour of the solution compatible with other unitarity triangle constraints [39].

Further details on *CP* violation in beauty and charm can be found in [55, 136].

69.4.3 *Charm Mixing and CP violation*

Time-dependent amplitude analyses in decays to final states that are accessible to both D^0 and $\overline{D}^0$ have unique sensitivity to mixing parameters. A Dalitz plot analysis of a self-conjugate final state, such as $K_S\pi^+\pi^-$ and $K_SK^+K^-$, allows the measurement of the phase difference between the relevant D^0 and $\overline{D}^0$ decay amplitudes, and thus a direct measurement of x and y, the normalized mass and width difference of the $D^0 - \overline{D}^0$ system's mass eigenstates. This is in contrast to decays like $D^0 \to K\pi$ [137] which only provide access to the decay-specific parameters x'^2, y'. The phase differences between D^0 and $\overline{D}^0$ amplitudes across the Dalitz plot affect these measurements in the same way as measurements of γ/ϕ_3 in $B^+ \to DK^+$, and can be taken into account in an amplitude model-independent way using the same charm threshold results [70, 77]. This approach recently resulted the first observation of the mass difference of the neutral charm mass eigenstates by LHCb [138], using a model-independent analysis of $D^0 \to K_S\pi^+\pi^-$. The "bin flip" technique applied in this measurements exploits that *CP*-conjugate regions of the $D^0 \to K_S\pi^+\pi^-$ Dalitz plot have near-identical experimental efficiencies [139]. This measurement is also sensitive to *CP* violation in mixing and in the interference between mixing and decay, which is discussed further in [55, 136]. The most precise values for charm mixing parameters result from a combination of charm data from LHCb and the charm threshold with input from γ-sensitive decays of the type $B^+ \to D^0K^+$ [68]. The substantial impact of the latter is noteworthy, as it improves the precision on y by a factor of two compared to the previous world average.

69.5 Summary

Multibody charm decays offer a rich phenomenology, including unique sensitivity to *CP* violation and charm mixing. This is a highly dynamic field with many new results (some of which we presented here) and rapidly increasing, high quality datasets. These datasets constitute a huge opportunity, but also a challenge to improve the theoretical descriptions of soft hadronic effects in multibody decays. For some measurements, model-independent methods, many relying on input from the charm threshold, provide a way of removing model-induced uncertainties. At the same time, substantial progress in the theoretical description of multibody decays is being made.

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